Lecture 10

7.3 - Trigonometric Substitution

We want to compute integrals involving the expressions:

① $\sqrt{a^2-x^2}$ ② $\sqrt{x^2+a^2}$ ③ $\sqrt{x^2-a^2}$

(where $a>0$).

The philosophy is to replace what's under the integral with something squared: y^2. Then $\sqrt{y^2}=|y|$. The trig identities

$$\sin^2\theta+\cos^2\theta=1 \quad \& \quad 1+\tan^2\theta=\sec^2\theta$$

fit this bill perfectly.

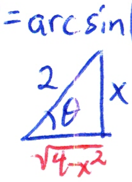
① Integrals involving $\sqrt{a^2-x^2}$

Since a is a constant, the identity $\sin^2\theta+\cos^2\theta=1$ works here: if we let $x=a\sin\theta$, then we get

$$\sqrt{a^2-x^2}=\sqrt{a^2-a^2\sin^2\theta}=\sqrt{a^2\cos^2\theta}=a|\cos\theta|$$

taking $-\frac{\pi}{2}\leq\theta\leq\frac{\pi}{2}$, we can remove the absolute value bars on $\cos\theta$.

Ex: Compute $\int \frac{x^2}{\sqrt{4-x^2}}\,dx$ $\quad x = 2\sin\theta \Rightarrow \theta = \arcsin\left(\frac{x}{2}\right)$, $\quad dx = 2\cos\theta\, d\theta$



$$\int \frac{x^2}{\sqrt{4-x^2}}\,dx = \int \frac{4\sin^2\theta}{\cancel{2\cos\theta}}(\cancel{2\cos\theta})\,d\theta = \int 4\sin^2\theta\, d\theta \overset{ⓐ}{=} \int 2(1-\cos 2\theta)\,d\theta$$

$$= 2\theta - \sin 2\theta + C \overset{ⓑ}{=} 2\theta - 2\sin\theta\cos\theta + C$$

$$= 2\arcsin\left(\frac{x}{2}\right) - \cancel{2}\left(\frac{x}{\cancel{2}}\right)\left(\frac{\sqrt{4-x^2}}{2}\right) + C = \boxed{2\arcsin\left(\frac{x}{2}\right) - \frac{1}{2}x\sqrt{4-x^2} + C}$$

ⓐ $\sin^2\theta = \frac{1-\cos 2\theta}{2}$

ⓑ $\sin 2\theta = 2\sin\theta\cos\theta$

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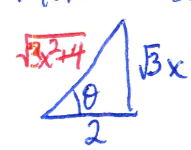
② Integrals involving $\sqrt{x^2+a^2}$

The identity $1+\tan^2\theta = \sec^2\theta$ is well suited to this: letting $x = a\tan\theta$, then

$$\sqrt{x^2+a^2} = \sqrt{a^2\tan^2\theta + a^2} = \sqrt{a^2\sec^2\theta} = a|\sec\theta|$$

Restricting θ to: $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, we can again drop the absolute value bars.

Ex: Compute $\int \frac{dx}{\sqrt{4+3x^2}} = \int \frac{dx}{\sqrt{4+(\sqrt{3}x)^2}}$ $\quad \sqrt{3}x = 2\tan\theta \longrightarrow \tan\theta = \frac{\sqrt{3}x}{2}$, $\quad dx = \frac{2}{\sqrt{3}}\sec^2\theta\, d\theta$



$$\int \frac{dx}{\sqrt{4+3x^2}} = \int \frac{\frac{2}{\sqrt{3}}\sec^2\theta}{2\sec\theta}\,d\theta = \frac{1}{\sqrt{3}}\int \sec\theta\, d\theta = \frac{1}{\sqrt{3}}\ln|\sec\theta + \tan\theta| + C$$

$$= \boxed{\frac{1}{\sqrt{3}}\ln\left|\frac{1}{2}\sqrt{3x^2+4} + \frac{\sqrt{3}}{2}x\right| + C}$$

3 Integrals involving $\sqrt{x^2-a^2}$

The identity $1+\tan^2\theta=\sec^2\theta$ is again useful here (write it $\sec^2\theta-1=\tan^2\theta$). Let $x=a\sec\theta$, then:

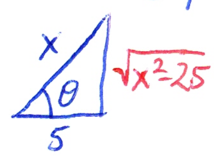
$$\sqrt{x^2-a^2}=\sqrt{a^2\sec^2\theta-a^2}=\sqrt{a^2\tan^2\theta}=a|\tan\theta|$$

To remove the absolute value bars, there are two good choices (the one you make is context dependent):

$$0\le\theta<\frac{\pi}{2} \quad \text{or} \quad \pi\le\theta<\frac{3\pi}{2}$$

Ex: Compute $\displaystyle\int\frac{dx}{x^2\sqrt{x^2-25}}$ $\quad x=5\sec\theta \to \theta=\operatorname{arcsec}\left(\frac{x}{5}\right)$

$dx=5\sec\theta\tan\theta\,d\theta$



$$\int\frac{dx}{x^2\sqrt{x^2-25}}=\int\frac{\cancel{5}\cancel{\sec\theta}\cancel{\tan\theta}}{(25\sec^{\cancel{2}}\theta)(\cancel{5}\cancel{\tan\theta})}\,d\theta=\int\frac{1}{25\sec\theta}\,d\theta$$

$$=\frac{1}{25}\int\cos\theta\,d\theta=\frac{1}{25}\sin\theta+C=\boxed{\frac{1}{25}\frac{\sqrt{x^2-25}}{x}+C}$$

Let's summarize this in a table:

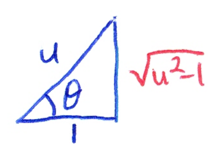
Expression	Substitution	θ-range	Trig Identity
$\sqrt{a^2-x^2}$	$x=a\sin\theta$ $\left(\theta=\arcsin\left(\frac{x}{a}\right)\right)$	$-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$	$\cos^2\theta=1-\sin^2\theta$
$\sqrt{a^2+x^2}$	$x=a\tan\theta$ $\left(\theta=\arctan\left(\frac{x}{a}\right)\right)$	$-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$	$1+\tan^2\theta=\sec^2\theta$
$\sqrt{x^2-a^2}$	$x=a\sec\theta$ $\left(\theta=\operatorname{arcsec}\left(\frac{x}{a}\right)\right)$	$0\le\theta<\frac{\pi}{2}$, or $\pi\le\theta<\frac{3\pi}{2}$	$\sec^2\theta-1=\tan^2\theta$

Sometimes, we can convert an integral into a trig sub integral by completing the square:

Ex: Compute $\int\sqrt{x^2+2x}\,dx$

$$\int\sqrt{x^2+2x}\,dx=\int\sqrt{(x^2+2x+1)-1}\,dx=\int\sqrt{(x+1)^2-1}\,dx \qquad \begin{array}{l}u=x+1\\ du=dx\end{array}$$

$$=\int\sqrt{u^2-1}\,du \quad (u=\sec\theta \quad du=\sec\theta\tan\theta\,d\theta)$$



$$=\int\tan\theta\,(\sec\theta\tan\theta\,d\theta)=\int\tan^2\theta\sec\theta\,d\theta=\int(\sec^3\theta-\sec\theta)\,d\theta$$

$$=\frac{\sec\theta\tan\theta}{2}+\frac{1}{2}\ln|\sec\theta+\tan\theta|-\ln|\sec\theta+\tan\theta|+C=\frac{1}{2}\left(u\sqrt{u^2-1}-\ln\left|u+\sqrt{u^2-1}\right|\right)+C$$

$$=\boxed{\frac{1}{2}\left[(x+1)\sqrt{(x+1)^2-1}-\ln\left[(x+1)+\sqrt{(x+1)^2-1}\right]+C\right.}$$

Further Examples

Ex: $\displaystyle\int \frac{dx}{x\sqrt{x^2-a^2}}$ $\quad x = a\sec\theta \longrightarrow \theta = \operatorname{arcsec}\left(\frac{x}{a}\right)$

$dx = a\sec\theta\tan\theta\,d\theta$

$$= \int \frac{\cancel{a\sec\theta\tan\theta}}{\cancel{a\sec\theta}\,(a\cancel{\tan\theta})}\,d\theta = \frac{1}{a}\int d\theta = \frac{1}{a}\theta + C$$

$$= \boxed{\frac{1}{a}\operatorname{arcsec}\left(\frac{x}{a}\right) + C}$$

Ex: $\displaystyle\int_0^3 \frac{t}{\sqrt{36-t^2}}\,dt$ $\quad t = 6\sin\theta$

$dt = 6\cos\theta\,d\theta$

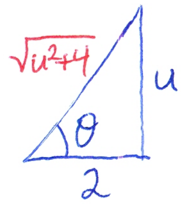
$\left(\begin{array}{l} 0 = 6\sin\theta \Rightarrow \theta = 0 \\ \left.\begin{array}{l} 3 = 6\sin\theta \\ \frac{1}{2} = \sin\theta \end{array}\right\} \Rightarrow \theta = \frac{\pi}{6} \end{array} \right)$

$$= \int_0^{\pi/6} \frac{6\sin\theta}{\cancel{6\cos\theta}}\left(\cancel{6\cos\theta}\,d\theta\right) = \int_0^{\pi/6} 6\sin\theta\,d\theta = -6\cos\theta\Big|_0^{\pi/6}$$

$$= -6\left(\frac{\sqrt{3}}{2} - 1\right) = \boxed{6 - 3\sqrt{3}}$$

Ex: $\displaystyle\int \frac{dv}{\sqrt{v^2-6v+13}} = \int \frac{dv}{\sqrt{(v^2-6v+9)+4}} = \int \frac{dv}{\sqrt{(v-3)^2+4}}$ $\quad u=v-3$, $du=dv$

$$= \int \frac{du}{\sqrt{u^2+4}} \quad \frac{u=2\tan\theta}{du=2\sec^2\theta\, d\theta} \quad \int \frac{\cancel{2}\sec^{\cancel{2}}\theta}{\cancel{2\sec\theta}}\, d\theta = \int \sec\theta\, d\theta$$



$$= \ln|\sec\theta+\tan\theta|+C$$

$$= \ln\left|\frac{1}{2}\sqrt{u^2+4}+\frac{u}{2}\right|+C$$

$$= \boxed{\ln\left|\frac{1}{2}\left(\sqrt{(v-3)^2+4}+(v-3)\right)\right|+C}$$

Ex: $\displaystyle\int_{\sqrt{2}/3}^{2/3} \frac{dx}{x^5\sqrt{9x^2-1}}$ $\quad 3x=\sec\theta \longrightarrow \theta=\operatorname{arcsec}(3x)$ $\quad$ Choose θ:

$dx=\frac{1}{3}\sec\theta\tan\theta\, d\theta$ $\quad 0\le\theta<\frac{\pi}{2}$

$$= \int_{\pi/4}^{\pi/3} \frac{\cancel{\tfrac{1}{3}}\cancel{\sec\theta}\cancel{\tan\theta}}{\frac{\sec^{\cancel{5}\,4}\theta}{3^{\cancel{5}\,4}}\cdot\cancel{\tan\theta}}\, d\theta$$

$\left(\begin{array}{l}\theta=\operatorname{arcsec}\left(3\left(\frac{2}{3}\right)\right) \Rightarrow \sec\theta=2 \Rightarrow \theta=\pi/3 \\ \theta=\operatorname{arcsec}\left(3\left(\frac{\sqrt{2}}{3}\right)\right) \Rightarrow \sec\theta=\sqrt{2} \Rightarrow \theta=\pi/4\end{array}\right)$

$$= \int_{\pi/4}^{\pi/3} \frac{1}{\frac{\sec^4\theta}{3^4}}\, d\theta = \int_{\pi/4}^{\pi/3} 3^4\cos^4\theta\, d\theta = 3^4\int_{\pi/4}^{\pi/3}\left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$$

$$= \frac{3^4}{2^2}\int_{\pi/4}^{\pi/3}\left(1+2\cos 2\theta+\cos^2 2\theta\right)d\theta = \frac{81}{4}\int_{\pi/4}^{\pi/3}\left(1+2\cos 2\theta+\frac{1}{2}+\frac{1}{2}\cos 4\theta\right)d\theta$$

$$= \frac{81}{4}\left(\frac{3\theta}{2}+\sin 2\theta+\frac{1}{8}\sin 4\theta\right)\Bigg|_{\pi/4}^{\pi/3} = \frac{81}{4}\left[\left(\frac{\pi}{2}+\frac{\sqrt{3}}{2}-\frac{1}{8}\frac{\sqrt{3}}{2}\right)-\left(\frac{3\pi}{8}+1+0\right)\right] = \boxed{\frac{81}{4}\left(\frac{\pi}{8}+\frac{7\sqrt{3}}{16}-1\right)}$$